FORBIDDEN SUBGRAPHS IN THE NORM GRAPH

SIMEON BALL AND VALENTINA PEPE

Abstract. We show that the norm graph with n vertices about $\frac{1}{2}n^{2-1/t}$ edges, which contains no copy of the complete bipartite graph $K_{t,(t-1)!+1}$, does not contain a copy of $K_{t+1,(t-1)!-1}$.

1. Introduction

Let H be a fixed graph. The *Turán number* of H, denoted $ex(n, H)$, is the maximum number of edges a graph with n vertices can have, which contains no copy of H. The Erdős-Stone theorem from [7] gives an asymptotic formula for the Turán number of any non-bipartite graph, and this formula depends on the chromatic number of the graph H.

When H is a complete bipartite graph, determining the Turán number is related to the "Zarankiewicz problem" (see [3], Chap. VI, Sect.2, and [9] for more details and references). In many cases even the question of determining the right order of magnitude for $ex(n, H)$ is not known.

Let $K_{t,s}$ denote the complete bipartite graph with t vertices in one class and s vertices in the other. The probabilistic lower bound for $K_{t,s}$

$$ex(n, K_{t,s}) \geqslant cn^{2-(s+t-2)/(st-1)}$$

is due to Erdős and Spencer [6]. Kővari, Sós and Turán [15] proved that for $s \geqslant t$

$$ex(n, K_{t,s}) \leqslant \tfrac{1}{2}(s-1)^{1/t}n^{2-1/t} + \tfrac{1}{2}(t-1)n. \tag{1.1}$$

The norm graph $\Gamma(t)$, which we will define the next section, has n vertices and about $\frac{1}{2}n^{2-1/t}$ edges. In [1] (based on results from [14]) it was proven that the graph $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$, thus proving that for $s \geqslant (t-1)!+1$,

$$ex(n, K_{t,s}) > cn^{2-1/t}$$

for some constant c.

Date: 10 March 2015.

The first author acknowledges the support of the project MTM2008-06620-C03-01 of the Spanish Ministry of Science and Education and the project 2014-SGR-1147 of the Catalan Research Council.

The second author acknowledges the support of the project "Decomposizione, proprietà estremali di grafi e combinatoria di polinomi ortogonali" of the SBAI Department of Sapienza University of Rome.

In [2], it was shown that $\Gamma(4)$ contains no copy of $K_{5,5}$, which improves on the probabilistic lower bound of Erdős and Spencer [6] for $ex(n, K_{5,5})$. In this article, we will generalise this result and prove that $\Gamma(t)$ contains no copy of $K_{t+1,(t-1)!-1}$. For $t \geqslant 5$, this does not improve the probabilistic lower bound of Erdős and Spencer, but, as far as we are aware, it is however the deterministic construction of a graph with n vertices containing no $K_{t+1,(t-1)!-1}$ with the most edges.

2. The norm graph

Suppose that $q = p^h$, where p is a prime, and denote by $\mathbb{F}_q$ the finite field with q elements. We will use the following properties of finite fields. For any $a, b \in \mathbb{F}_q$, $(a+b)^{p^i} = a^{p^i} + b^{p^i}$, for any $i \in \mathbb{N}$. For all $a \in \mathbb{F}_{q^i}$, $a^q = a$ if and only if $a \in \mathbb{F}_q$. Finally $N(a) = a^{1+q+\cdots+q^{k-1}} \in \mathbb{F}_q$, for all $a \in \mathbb{F}_{q^k}$, since $N(a)^q = N(a)$.

Let $\mathbb{F}$ denote an arbitrary field. We denote by $\mathbb{P}_n(\mathbb{F})$ the projective space arising from the $(n+1)$-dimensional vector space over $\mathbb{F}$. Throughout dim will refer to projective dimension. A point of $\mathbb{P}_n(\mathbb{F})$ (which is a one-dimensional subspace of the vector space) will often be written as $\langle u \rangle$, where u is a vector in the $(n+1)$-dimensional vector space over $\mathbb{F}$.

Let $\Gamma(t)$ be the graph with vertices $(a, \alpha) \in \mathbb{F}_{q^{t-1}} \times \mathbb{F}_q$, $\alpha \neq 0$, where (a, α) is joined to (a', α') if and only if $N(a + a') = \alpha\alpha'$. The graph $\Gamma(t)$ was constructed in [14], where it was shown to contain no copy of $K_{t,t!+1}$. In [1] Alon, Rónyai and Szabó proved that $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$. Our aim here is to show that it also contains no $K_{t+1,(t-1)!-1}$, generalizing the same result for $t = 5$ presented in [2].

Let
$$V = \{(1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}}) \mid a \in \mathbb{F}_{q^{t-1}}\} \subset \mathbb{P}_{2^{t-1}-1}(\mathbb{F}_{q^{t-1}}).$$

The set V is the affine part of an algebraic variety that is in turn a subvariety of the Segre variety
$$\Sigma = \underbrace{\mathbb{P}_1 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1}_{t-1 \text{ times}},$$
where $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{F}_q)$. We briefly recall that a Segre variety is the image of the Segre embedding:

$$\sigma : (v_1, v_2, \ldots, v_k) \in \mathbb{P}_{n_1-1} \times \mathbb{P}_{n_2-1} \times \cdots \times \mathbb{P}_{n_k-1} \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_k \in \mathbb{P}_{n_1 n_2 \cdots n_k - 1}$$

i.e. it is the set of points corresponding to the simple tensors. For the reader that is not familiar to tensor products we remark that, up to a suitable choice of coordinates, if $v_i = (x_0^{(i)}, x_1^{(i)}, \ldots, x_{n_i-1}^{i})$, then $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ is the vector of all possible products of type: $x_{j_1}^{(1)} x_{j_2}^{(2)} \cdots x_{j_k}^{(k)}$ (see [12] for an easy overview on Segre varieties over finite fields).

Then, the affine point $P_a = (1,a) \otimes (1,a^q) \otimes \cdots \otimes (1,a^{q^{t-2}})$ has coordinates indexed by the subsets of $T := \{0,1,\dots,t-1\}$, where the S-coordinate is

$$(\prod_{i \in S} a^{q^i}),$$

for any non-empty subset S of T and

$$\prod_{i \in S} a^{q^i} = 1$$

when $S = \emptyset$ (see [16]).

Let $n = 2^{t-1} - 1$.

We order the coordinates of $\mathbb{P}_n(\mathbb{F}_{q^{t-1}})$ so that if the i-th coordinate corresponds to the subset S, then the $(n-i)$–th coordinate corresponds to the subset $T \setminus S$.

Embed the $\mathbb{P}_n(\mathbb{F}_{q^{t-1}})$ containing V as a hyperplane section of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$ defined by the equation $x_{n+1} = 0$.

Let b be the symmetric bilinear form on the $(n+2)$-dimensional vector space over $\mathbb{F}_{q^{t-1}}$ defined by

$$b(u,v) = \sum_{i=0}^{n} u_i v_{n-i} - u_{n+1} v_{n+1}.$$

Let $\perp$ be defined in the usual way, so that given a subspace Π of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$, $\Pi^\perp$ is the subspace of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$ defined by

$$\Pi^\perp = \{v \mid b(u,v) = 0, \text{ for all } u \in \Pi\}.$$

We wish to define the same graph $\Gamma(t)$, so that adjacency is given by the bilinear form. Let $P = (0,0,0,\dots,1)$. Let Γ' be a graph with vertex set the set of points on the lines joining the aff points of V to P obtained using only scalars in $\mathbb{F}_q$, distinct from P and not contained in the hyperplane $x_{n+1} = 0$. Join two vertices $\langle u \rangle$ and $\langle u' \rangle$ in Γ' if and only if $b(u,u') = 0$. It is a simple matter to verify that the graph Γ' is isomorphic to the graph $\Gamma(t)$ by the map $P_a + \alpha P \mapsto (a,\alpha)$ since

$$N(a+b) - \alpha\beta = \sum_{S \subseteq T} \prod_{i \in S,\ j \in T \setminus S} a^{q^i} b^{q^j} - \alpha\beta = b(u,v),$$

where

$$u = (1,a) \otimes (1,a^q) \otimes \cdots \otimes (1,a^{q^{t-2}}) + \alpha P,$$

and

$$v = (1,b) \otimes (1,b^q) \otimes \cdots \otimes (1,b^{q^{t-2}}) + \beta P.$$

We shall refer to Γ' as $\Gamma(t)$ from now on.

We recall some known properties of Σ and its subvariety

$$\mathcal{V} = \{(a,b) \otimes (a^q,b^q) \otimes \cdots (a^{q^{t-2}}, b^{q^{t-2}}) \mid (a,b) \in \mathbb{P}_1(\mathbb{F}_{q^{t-1}})\}$$

and prove a new one in Theorem 2.5.

Let $\overline{\mathbb{F}_q}$ denote the algebraic closure of $\mathbb{F}_q$ and consider Σ as the Segre variety over $\overline{\mathbb{F}_q}$.

THEOREM 2.1. *Σ is a smooth irreducible variety.*

THEOREM 2.2. *The dimension of Σ (as algebraic variety) is $t-1$ and its degree is $(t-1)!$.*

Proof. The (Segre) product $X \times Y$ of two varieties X and Y of dimension d and e has dimension $d+e$, see, for example [13], page 138. The Hilbert polynomial of $X \times Y$ is the product of the Hilbert polynomials of X and Y (see [13, Chapter 18]). The Hilbert polynomial $h(m)$ of $\mathbb{P}_1$ is $m+1$, hence the Hilbert polynomial of $\Sigma = \underbrace{\mathbb{P}_1 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1}_{t-1 \text{ times}}$ is $h_\Sigma(m) = (m+1)^{t-1}$. Since the leading term of h_Σ is 1 and the dimension of Σ is $t-1$, we have that the degree of Σ is $(t-1)!$. □

THEOREM 2.3. [16] *Any t points of $\mathcal{V}$ are in general position.*

THEOREM 2.4. [11] *If $t+1$ points span a $(t-1)$-dimensional projective space, then that space contains $q+1$ points of $\mathcal{V}$.*

THEOREM 2.5. *If a subspace of codimension t contains a finite number of points of Σ then it contains at most $(t-1)!-2$ points of Σ.*

Proof. By Theorem 2.1, Σ is smooth, so it is regular at each of its points, i.e., if $T_P\Sigma$ is the tangent space of Σ at a point $P \in \Sigma$, then $\dim T_P\Sigma = t-1$.

Let Π be a subspace of codimension t containing a finite number of points of Σ. Let $P \in \Pi \cap \Sigma$. Then $\dim\langle T_P\Sigma, \Pi\rangle \leqslant n-1$. Therefore, there is a hyperplane H containing $\langle T_P\Sigma, \Pi\rangle$.

Suppose that H contains another tangent space $T_R\Sigma$, with $R \in \Pi \cap \Sigma$. The algebraic variety $H \cap \Sigma$ has dimension $t-2$ (since Σ is irreducible) and it has two singular points, P and R. Since $\dim H \cap \Sigma = t-2$ as an algebraic variety, there must be a linear subspace Π_1 of codimension $t-2$ in H containing Π and such that $\Pi_1 \cap H \cap \Sigma$ consists of $\deg(H \cap \Sigma) \leqslant (t-1)!$ points of Σ counted with their multiplicity. Since Π_1 contains P and R, which are singular points and so with multiplicity at least 2, we have that

$$|\Pi \cap \Sigma| \leqslant |\Pi_1 \cap \Sigma| \leqslant (t-1)! - 2.$$

Suppose now that H does not contain any other tangent space $T_R\Sigma$ with $R \in \Pi \cap \Sigma$, $R \neq P$. Then take $R \in \Pi \cap \Sigma$ and consider a hyperplane $H' \neq H$ containing $\langle T_R\Sigma, \Pi\rangle$. Then the tangent spaces of P and R with respect to $H \cap H' \cap \Sigma$ are $T_P\Sigma \cap H'$ and $T_R\Sigma \cap H$, and they both have dimension $t-2$ (as linear spaces).

If $\dim H \cap H' \cap \Sigma = t-3$ as an algebraic variety, then P and R are two singular points of $H \cap H' \cap \Sigma$ and we can find, as before, a linear subspace Π_1 of codimension $t-3$ in $H \cap H'$ such that it contains Π and intersects $H \cap H' \cap \Sigma$ in $\deg(H \cap H' \cap \Sigma) \leqslant (t-1)!$

points, counted with their multiplicity. Since P and R have multiplicity at least 2, we have

$$|\Pi \cap \Sigma| \leqslant |\Pi_1 \cap \Sigma| \leqslant (t-1)! - 2.$$

If $\dim H \cap H' \cap \Sigma = t-2$ as an algebraic variety, then $H \cap \Sigma$ is reducible. Hence, we have

$$H \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_r,$$

where $\mathcal{V}_i$ is an irreducible variety of dimension $t-2$, for all $i = 1, \ldots, r$. So we have

$$H \cap H' \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r,$$

where $\mathcal{W}_i$ is a hyperplane section of $\mathcal{V}_i$, for all $i = s+1, \ldots, r$. We observe that also $H' \cap \Sigma$ has to be reducible and, since the decomposition in irreducible components is unique, we have

$$H' \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{V}'_{s+1} \cup \mathcal{V}'_{s+2} \cup \cdots \cup \mathcal{V}'_r,$$

where $\mathcal{V}_i$ and $\mathcal{V}'_j$ are irreducible varieties of dimension $t-2$.

We have, by hypothesis, that $T_P\Sigma \subset H$ and $P \in \Pi$. So either $P \in \mathcal{V}_i$ and it is singular for $\mathcal{V}_i$, for some $i \in \{1, 2, \ldots, r\}$, or it is not singular for $\mathcal{V}_\ell$, for any $\ell \in \{1, 2, \ldots, r\}$.

Suppose we are in the first case. We know that $P \in \Pi \subset H'$. If $\mathcal{V}_i \subseteq H'$, then P is singular for an irreducible component of $H' \cap \Sigma$ and so $T_P\Sigma \subset H'$, contradicting our hypothesis, so $\mathcal{V}_i$ is not contained in H' and $H' \cap \mathcal{V}_i = \mathcal{W}_i$. We have that $\dim T_P\Sigma \cap H' = t-2$ (as linear subspace) and $\dim \mathcal{W}_i = t-3$ (as algebraic variety), so P is singular for $\mathcal{W}_i$.

Suppose now that P is not singular for any $\mathcal{V}_i$, so the dimension of $T_P\mathcal{V}_i$, as a subspace, is $t-2$. If $P \notin \mathcal{V}_j$, for any $i \neq j$, then

$$T_P(H \cap \Sigma) = T_P(\mathcal{V}_i) = T_P(\Sigma),$$

a contradiction since the dimension of $T_P(\Sigma)$ is $t-1$. Hence $P \in \mathcal{V}_i \cap \mathcal{V}_j$, and so P is contained in the intersection of two components of $H' \cap \Sigma$, so it is again a singular (or multiple) point. The same is true for the point R such that $T_R\Sigma \subset H'$, so in

$$\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r$$

there are at least two multiple points and when we sum up all the degrees, we count at least two points twice, hence, by

$$\sum_{i=1}^{s} \deg \mathcal{V}_i + \sum_{j=s+1}^{r} \deg \mathcal{W}_j \leqslant (t-1)!,$$

we get that the number of points in

$$\Pi \cap (\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r),$$

is at most $(t-1)! - 2$. $\square$

Remark One could wonder whether one could try with one more hyperplane H'' such that $T_Q\Sigma \subset H''$, $T_Q\Sigma \not\subseteq H$, $T_Q\Sigma \not\subseteq H$' and $Q \in \Pi$. However, it can happen that $H \cap H' \cap H'' = H \cap H'$, so $\dim T_Q\Sigma \cap H \cap H' \cap H'' = t-2$ (as a linear space) and $\dim H \cap H' \cap H'' \cap \Sigma = t-2$, so Q would not be a singular point of

$$H \cap H' \cap H'' \cap \Sigma = H \cap H' \cap \Sigma.$$

The locus of hyperplanes containing a tangent space to a variety X of $\mathbb{P}^n$ is a variety X^* of the dual space $(\mathbb{P}_n)^*$ (see, e.g., [13, Chapter 15]). Let Σ^* be the dual variety of Σ. From [17], we know that Σ^* is a hypersurface, hence, if d is the degree of Σ^*, then the number of points of Σ^* on a general line of $(\mathbb{P}_n)^*$ is d. Suppose that the line of $(\mathbb{P}_n)^*$ defined by $H \cap H'$ is general, hence if $|\Pi \cap \Sigma| > d$, then we could find a point $Q \in \Pi \cap \Sigma$ such that $T_Q\Sigma \subset H''$ and H'' is a hyperplane not containing $H \cap H'$. If $d > (t-1)! - 2$ then we would not be able to get a better bound than the bound in Theorem 2.5. The degree of Σ^* is found in [10], where it is given by N_{t-1}, where N_r is defined by the generating function

$$\sum_{r \geqslant 0} N_r \frac{z^r}{r!} = \frac{e^{-2z}}{(1-z)^2}.$$

Hence $d = \deg \Sigma^*$, is the evaluation of

$$\left(\frac{e^{-2z}}{(1-z)^2} \right)^{(t-1)}$$

at $z = 0$, where we denote by $f^{(n)}$ the n–th derivative of the function f.

Let $F = fg$, where f and g are two functions, then

$$F^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(i)} g^{(n-i)}.$$

Let

$$f = e^{-2z} \text{ and } g = (1-z)^{-2}.$$

It is easy to see that

$$f^{(i)} = (-2)^i f \text{ and } g^{(i)} = (i+1)!(1-z)^{-(i+2)}.$$

Since $f(0) = 1$, we have that $F^{(n)}$, evaluated at $z = 0$, is

$$\sum_{i=0}^{n} \binom{n}{i} (-2)^i (n+1-i)!.$$

When $n = t-1$ and we have

$$d = N_{t-1} = \sum_{i=0}^{t-1} \binom{t-1}{i} (-2)^i (t-i)!.$$

Now

$$\sum_{i=0}^{t-1} \binom{t-1}{i} (-2)^i (t-i)! = (t-1)! \sum_{i=0}^{t-1} \frac{(-2)^i}{i!}(t-i).$$

Note that

$$\sum_{i=0}^{t-1} \frac{(-2)^i}{i!}(t-i) = 1$$

for $t = 5$ and

$$\sum_{i=0}^{t} \frac{(-2)^i}{i!}(t+1-i) - \sum_{i=0}^{t-1} \frac{(-2)^i}{i!}(t-i) = \sum_{i=0}^{t} \frac{(-2)^i}{i!}.$$

Since $\sum_{i=0}^{5} \frac{(-2)^i}{i!} = \frac{1}{15}$ and

$$\frac{(-2)^{n-1}}{(n-1)!} - \frac{(-2)^n}{n!} = \frac{2^{n-1}(n-2)}{n!} > 0$$

when $n \geqslant 3$ is odd,

$$\sum_{i=0}^{t} \frac{(-2)^i}{i!} > 0$$

for all $t \geqslant 4$ and so

$$\sum_{i=0}^{t-1} \frac{(-2)^i}{i!}(t-i)$$

is an increasing function. Thus, for $t \geqslant 5$,

$$\sum_{i=0}^{t-1} \frac{(-2)^i}{i!}(t-i) \geqslant 1,$$

and so

$$(t-1)! \sum_{i=0}^{t-1} \frac{(-2)^i}{i!}(t-i) \geqslant (t-1)!$$

and hence $d = N_{t-1} > (t-1)! - 2$.

Theorem 2.6. *For $q \geqslant (t-1)! + 1$ the graph $\Gamma(t)$ contains no $K_{t+1,(t-1)!-1}$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_{t+1}\}$ be $t+1$ distinct vertices of $\Gamma(t)$. The set of common neighbours of the elements of X is $\Pi^\perp \cap \Gamma(t)$, where Π is the subspace spanned by X. If any two elements of X project from P onto the same point of V, then $P \in \Pi$ and hence $\Pi^\perp \subset P^\perp$. Since $P^\perp$ is the hyperplane $x_{n+1} = 0$, $\Pi^\perp \cap \Gamma(t) = \emptyset$, and the elements of X have no common neighbour.

Therefore, we assume now that all the points in X project from P onto distinct points of V. Then, by Theorem 2.3, $\dim \Pi \geqslant t-1$.

If $\dim \Pi = t-1$, then by Theorem 2.4, the projection of Π onto V contains at least q points of V (we recall that V is the affine part of $\mathcal{V}$ and the hyperplane section we removed contains just one point of $\mathcal{V}$). Therefore, there are at least q points Y of Π on the lines joining P to the points of V. We wish to prove that the points of Y are vertices of the graph $\Gamma(t)$. To do this, we have to show that the points of Y, which are of the form $\langle (v,\lambda)\rangle$, where $v \in V$ and $\lambda \in \overline{\mathbb{F}_q}$, are of the form $\langle (v,\lambda)\rangle$, where $v \in V$ and $\lambda \in \mathbb{F}_q$. Assuming that the vertices in X have at least two common neighbours, we can suppose that there is a common neighbour of the elements of X of the form $\langle (u,\mu)\rangle$, where $u \in V$, $u \neq -v$ and $\mu \in \mathbb{F}_q$, is a common neighbour of the elements of X. Then $\langle (u,\mu)\rangle$ is in $\Pi^\perp$ and since $Y \subset \Pi$,

$$N(u+v) = \lambda\mu.$$

Since $N(u+v) \in \mathbb{F}_q$ and $\mu \in \mathbb{F}_q$, we have that $\lambda \in \mathbb{F}_q$ and so the points of Y are vertices of the graph $\Gamma(t)$. Therefore, the vertices of X have at least q common neighbours. Since Γ contains no $K_{t,(t-1)!+1}$, if $q \geqslant (t-1)!+1$, then this case cannot occur.

If $\dim \Pi = t$ then $\dim \Pi^\perp = n-t$. Let Y be the points of $\Pi^\perp$ which project from P onto V. Arguing as in the previous paragraph, the points Y are vertices of the graph $\Gamma(t)$. Since the vertices of X have at most $(t-1)!$ common neighbours, there are a finite number of points in Y and so a finite number of points in the projection of $\Pi^\perp$ onto V. By Theorem 2.5, this projection contains at most $(t-1)!-2$ points of V, so there are at most $(t-1)!-2$ points in Y. Therefore, the vertices in X have at most $(t-1)!-2$ common neighbours. □

References

[1] N. Alon, L. Rónyai and T. Szabó, Norm-Graphs: Variations and applications, *J. Combin. Theory Ser. B*, **76** (1999) 280–290.
[2] S. Ball and V. Pepe, Asymptotic improvements to the lower bound of certain bipartite Turán numbers, *Combin. Probab. Comput.*, **21** (2012) 323–329.
[3] B. Bollobás, *Extremal Graph Theory*, Academic Press, San Diego, 1978.
[4] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.*, **9** (1966) 281–289.
[5] P. Erdős, A. Rényi and V. T. Sós, On a problem of graph theory, *Studia, Sci. Math. Hungar.*, **1** (1966) 215–235.
[6] P. Erdős and J. Spencer, *Probabilistic Methods in Combinatorics*, Academic Press, London, New York, Akadémiai Kiadó, Budapest, 1974.
[7] P. Erdős and A. H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.*, **52** (1946) 1087–1091.
[8] Z. Füredi, An upper bound on Zarankiewicz' problem, *Combin. Probab. Comput.*, **5** (1996) 29–33.
[9] Z. Füredi, New asymptotics for bipartite Turán numbers, *J. Combin. Theory Ser. A*, **75** (1996) 141–144.
[10] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinski, Hyperdeterminants, *Adv. Math.*, 84 (1990), pp. 237–254.
[11] L. Giuzzi and V. Pepe, Families of twisted tensor product codes, *Designs Codes Cryptogr.*, **67** (2013) 375–384.

[12] J.W.P. Hirschfeld and J.A. Thas, *General galois geometries*, Oxford University Press, New York (NY), 1991.
[13] J. Harris, *Algebraic Geometry. A first course*, Graduate Texts in Mathematics, **133**, Springer-Verlag, New York, 1992.
[14] J. Kollár, L. Rónyai and T. Szabó, Norm-graphs and bipartite Turán numbers, *Combinatorica*, **16** (1996) 399–406.
[15] T. Kővári, V. T. Sós and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.*, **3** (1954) 50–57.
[16] V. Pepe, On the algebraic variety $\mathcal{V}_{r,t}$, *Finite Fields Appl.*, 17 (2011) 343–349.
[17] J. Weyman and A. Zelevinsky, Singularities of Hyperdeterminants, *Ann. Inst. Fourier*, **46:3** (1996), 591-644.

Simeon Ball
Departament de Matemàtica Aplicada IV,
Universitat Politècnica de Catalunya, Jordi Girona 1-3, Mòdul C3, Campus Nord,
08034 Barcelona, Spain
simeon@ma4.upc.edu

Valentina Pepe
SBAI Department,
Sapienza University of Rome, Via Antonio Scarpa 16
00161 Rome, Italy
valepepe@sbai.uniroma1.it